RANDOMIZING WHEN TIME IS NOT WELL-ORDERED

BY

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ABSTRACT

This is a sequel to a previous work by the author, and contains the following result, announced therein: if the set of times at which a player in a game is called upon to move contains an infinite decreasing sequence, an appropriate reformulation of the Kuhn-Aumann-Wald-Wolfowitz Theorem on the equivalence of different ways of randomizing is no longer true.

1. Pure strategies when time is not well ordered

A game in which time runs over the negative integers does not appear to be a very natural construct. If however, we consider the fact that only the order structure of the time-values is relevant, games with negative integer time appear as a necessary first step in the study of games in positive real time, since the latter contains the former as an ordered subset.

In Game Theory, a "strategy" for a player is essentially a rule that tells the player, whenever it is this turn to move, where to move, as a function of what he knows at this stage. Even on this informal level, this notion of strategy is not appropriate for a game where the times when the player's turn comes, include an infinite decreasing sequence, such as the times $\{\cdots, \frac{1}{3}, \frac{1}{2}, 1\}$. For consider a game where the player remembers all his past moves. The instruction "Do what you did last time", though well defined at every stage, does not determine the player's moves uniquely; rather, it is compatible with all constant sequences of moves. The instruction "Do X, if you've never done it before. Otherwise, do Y" is not compatible with any sequence, though it too is well defined at any stage. In the language of [4], the *effect* of the first instruction is not unique, and for the second instruction, no effect can be defined. To establish a formal framework for these considerations, we adopt an adaptation of Aumann's concept of a "game normalized for all players but one" [1] to the special case of "perfect recall," and define a *game* as follows:

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(1) θ, the set of moving times, is either the set Z of all integers, or the Z^+ of all positive integers;

(2) At time $i \in \theta$, the player is "told" a number z_i. He "recalls" everything he was told earlier, and everything he did earlier.

A (pure) strategy for the player is defined in games where $\theta = Z^+$ as a sequence m of measurable functions m_i, defining the player's moves y_i inductively as $y_i = m(z_1, y_1, z_2, y_2, \cdots, z_{i-1}, y_{i-1}, z_i)$. The *effect* of m is the mapping from the set of z-sequences to the set of y-sequences hereby defined.

Since the sequence m fails to determine an effect when $\theta = Z$, it cannot be considered a strategy. We rename it a "strategic recursion". Rather than try and redefine "strategy" appropriately, one may start out from mappings of all z-sequences into y-sequences, and ask which mappings should be considered available to the player. One reasonable requirement on such a mapping is that it satisfy a strategic recursion m, that is, we require that for some sequence of (measurable) functions m_i, the sequence (z_i) and its image under the mapping (y_i) satisfy the equations $y_i = m_i(\cdots, z_{i-2}, y_{i-2}, z_{i-1}, y_{i-1}, z_i)$. This requirement amounts to each y_i being determinable at the right time by the player, from what he then knows. For $\theta = Z^+$, this requirement characterizes the effects of strategies. When θ includes the negative integers Z^-, starting out from a mapping, and requiring that it satisfy the recursion m eliminates the possibility of undefined or not uniquely-defined mappings. However, assuming that any mapping that satisfies a strategic recursion is available to the player, leads for $\theta \supset Z$ to a new difficulty: Consider the recusion $m_i = y_{i-1} + z_i (\mathrm{mod}\, 2)$, defined for sequences of zeros and ones, indexed by negative integers. The equation

$$y_i = z_{i+1} + \cdots + z_{-1} (\mathrm{mod}\, 2)$$

defines a mapping from z-sequences into y-sequences that satisfies the recursion m_i. The player could calculate y_i at every stage, using just z_i and his own previous move. But for $i = -2$ the recursion yields $y_{-2} = z_{-1}$, so by the time the player has been told z_{-2}, and calculated y_{-2}, he can predict z_{-1} at a time when the opponents should still be free to choose z_{-1} arbitrarily from the set $\{0, 1\}$. This "time machine paradox" should be ruled out by adding the following requirement for a mapping to be available to the player:

(*) *For every i, two z-sequences that agree till i are assigned y-sequences that agree till i.*

When $\theta = Z^+$, this requirement is equivalent to the existence of a recursion, without its measurability. When $\theta \supset Z^-$, it still implies a recursion, but is not implied by it, as the example has shown, even when the recursion itself is measurable. In fact, for $\theta = Z$ the innocuous looking, and obviously measurable, recursion $y_i = z_i + y_{i-1} \pmod 2$ has the property that ***every mapping compatible with it either leads to the time-machine paradox, or is not Lebesgue measurable.***

PROOF. For every z-sequence, the recursion has two solutions. For a mapping from z-sequences to y-sequences to satisfy the recursion, its range A must include exactly one of each such pair of solutions. Belonging to A either is a tail property ($i \to -\infty$) or it isn't. If it isn't, there exists a sequence (y_i) in A that shares an initial tail with a sequence not in A. Continuing along the sequence (y_i) beyond this initial tail, there must be one index j such that the values $\cdots, y_j$ and the fact $(y_i) \in A$ determine the value of y_{j+1}. But then they also determine $z_{j+1} = y_j + y_{j+1} \pmod 2$, and the time-machine paradox occurs. On the other hand, if A is a tail set, its measurability would imply that its measure (Lebesgue measure, when the sequence is interpreted as a binary fraction) would be 0 or 1; however, changing each y_i into $1 - y_i$ maps A onto its complement, and leaves Lebesgue measure invariant, implying that A could only have the measure $\frac{1}{2}$. Q. E. D.

A nonmeasurable mapping, compatible with the recursion, that has property (*), and therefore does not lead to a time-machine-paradox, can be obtained as follows: Pair each sequence (y_i) with the complementary sequence, pass to tail--equivalence-classes, use the axiom of choice to pick one element from each pair, and let A be the set of all equivalence classes that were picked. The mapping that sends each z-sequence into the unique y-sequence that is compatible with it under the recursion, and whose tail-class is in A, cannot be measurable, but it does have property (*).

Let the payoff be defined so that the player "wins" if the recursion holds. Then there is no winning strategy for the player, if a (pure) ***strategy*** is a measurable mapping that has property (*); if nonmeasurable mappings are permitted, the player can make sure to win.

2. Randomization

A mixed, or ***random strategy***, can be defined in the present setting as a measurable function on the product of a standard space (unit interval with Lebesgue measure) with the space of all z-sequences, into the space of all y-sequences, such that for each element of the standard space a (pure) strategy is obtained. The (random) ***effect*** of such a strategy is the function that associates with each z-sequence the distribution of the y-sequence induced by the given random strategy.

We shall not define randomization strategy the way it was done in [4], that is, as a "random recursion". Even in the special case of pure strategies, we have seen that when θ includes the negative integers such a definition is not appropriate. Rather, we define an ***admissible effect*** as a measurable mapping from the z-sequences into the set of probability measures on the set of y-sequences, for which the following random analogue of (*) holds:

(**) ***For every i, z-sequences that agree until i are assigned measures whose projections on*** $(\cdots, y_{i-1}, y_i)$ ***agree.***

The (convex) set of all admissible effects contains the (convex) set of all effects of random strategies. The pure strategies are extreme points of either set. They are the only extreme points of the latter set; however, if θ includes the negative integers, ***there exists an admissible effect that is not the effect of any random strategy.***

PROOF. For every z-sequence, assign the measure $\frac{1}{2}$ each to the two y-sequences compatible with it under the recursion $y_i = y_{i-1} + z_i \pmod 2$. Property (**) is easily checked, and so is the measurability of this mapping. Hence it defines an admissible effect. However, according to what was established above, no measurable pure strategy solves that recursion, and if there were a random strategy with the given effect, almost all of its pure strategies would be solutions of the recursion. Q. E. D.

Thus, when the moves of a game are not well-ordered in time, the equivalence of different ways of randomizing fails to hold. Note that the existence of an effect that is achievable by the player, while it is not attainable by mixing pure strategies, violates the equivalence in a direction opposite to the direction in which the equivalence is violated by a game that is not of perfect recall. In the latter, it is by mixing pure strategies that an effect not available otherwise is attained.

REMARK. The conditional distribution of the y-sequence, given the z-sequence in the example that yields the proof above, is essentially the same conditional distribution as the one ocurring in the counterexample in [2]. It was, in fact, a closer analysis of the latter that yielded the idea for this paper.

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